Promoting the understanding of mathematics in physics at secondary level

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ABSTRACT This article explores some of the common mathematical difficulties that 11- to 16-year-old students experience with respect to their learning of physics. The definition of *understanding* expressed in the article is in the sense of *transferability* of mathematical skills from topic to topic within physics as well as between the separate sciences and mathematics. It is argued that students who are taught the reasoning behind the processes are less likely to compartmentalise their learning. Some strategies, particularly concerning the language of mathematics, are discussed.

What is it to understand the mathematics?

'Knowledge' and 'understanding' are words that are so ubiquitous in education that it is easy to lose sight of their individual meanings and of their importance to learning. There are many possible meanings for these words depending on the context and many learned authors have written about just that. However, my intention here is to explore the importance of 'understanding' when learning mathematical aspects of physics (and science in general).

Let us first take the meaning of 'know' to be 'is able to recall' and then take the following description of 'understanding' for the discussion that follows:

> ... *understanding in mathematics implies an ability to recognise and make use of a mathematical concept in a variety of settings, including some which are not immediately familiar.* (Cockcroft, 1982: 68)

Consider the following, which is intended to exemplify the difference between knowledge and understanding, not just for the purposes of discussion in this article but also as an example to be used with students. The programme of study for 11- to 14-year-olds in England states that children should be taught to calculate and solve problems involving areas of circles (Department for Education, 2013a: 8). The mathematics subject content for 14- to 16-year-olds in England states that students should know the formula for finding the area of a circle and be able to calculate areas of circles (Department for Education, 2013b: 10). The intention here is for students to *understand why* a single formula can be applied to all circles whatever their size.

It is a simple exercise to demonstrate. Students draw a circle using compasses, choosing the radius of the circle themselves. They then cut a piece of string whose length is equal to the diameter. The string is laid out carefully along the circumference and the number of times that it fits along the whole circumference is measured (Figure 1). (This can also be a useful exercise in ratios as the remainder will need to be calculated as a proportion of the length of the string.)

The students will still only 'know' that the number of times the diameter fits around the circumference is around 3.14, a number we call π, but, by comparing their results to those of the rest of the class whose circles will have been different sizes, they can come to 'understand' that the ratio of circumference to diameter is always 3.14 : 1. They can see that what they have come to know about one circle is transferable to all circles. From here, students can go on to see how the formula for the area of a circle can be demonstrated and that the formula applies to all circles, whatever their size (Figure 2).

It could be argued that once students have followed this method for showing that the area

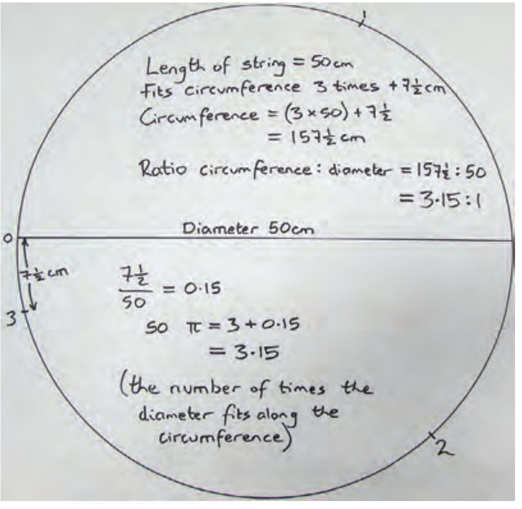

Figure 1 Finding the ratio of circumference to diameter. The string is cut to the length of the diameter of the circle and then carefully laid around the circumference. Students calculate how many times the string fits around the circumference and no matter how big their circles are it will always be approximately 3.14.

of a circle is πr^2 they will still not necessarily 'understand' but simply 'know' a method for demonstrating the fact. However, it is not the specific process that is important; it is the fact that the rule can be applied to all circles of all sizes that is crucial.

If students are to apply a single mathematical principle in a variety of contexts, they must understand the universality of the principle. This is such an important point but it is often lost on students, who tend to compartmentalise their learning. The phrase 'transferable' is given to skills that can be applied across a variety of subjects and contexts and teachers are often frustrated when they hear that students 'can do it' in maths but 'can't do it' in physics, or 'could do it when we studied energy but can't do it with forces'.

My argument is that focusing on *understanding the transferability* of the skills, rather than just on the processes that will get the right answer in an isolated context, is crucial to student progress.

Let us take the example of percentages. Students will use percentages in many non-science subjects as well as maths and of course all three sciences. Looking at a specific example from physics, students might well be asked to calculate the amount of thermal energy lost from a house through the windows as a percentage of the total amount of energy lost in a specified time. Giving the students a *process* for this particular problem will allow them to succeed in this specific context:

1. Find the total energy lost by all methods from the house.
2. Divide the energy lost through the windows by the total energy lost through the whole house.
3. Multiply your answer by 100 and don't forget to include the % symbol.

This is fine to an extent but it assumes that either the students already understand what a percentage is or, worse, it implies that students do not need to understand percentages provided that they can follow this process and get the right

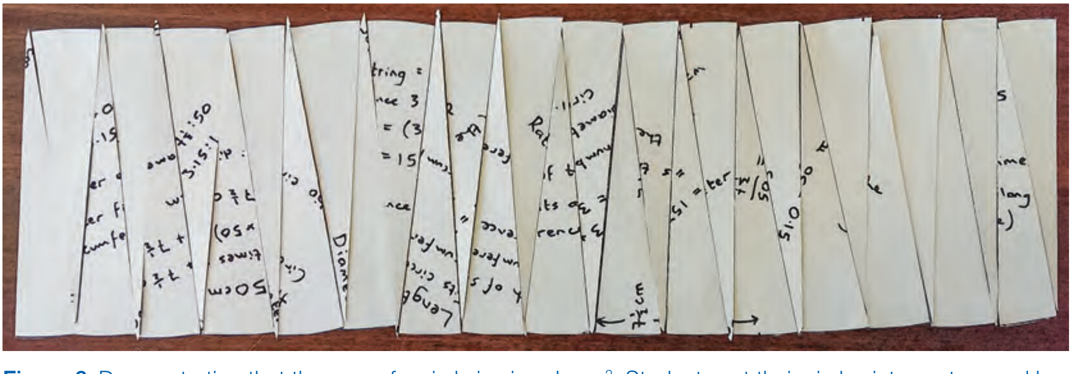

Figure 2 Demonstrating that the area of a circle is given by πr^2. Students cut their circles into sectors and lay them out as shown. The smaller the sectors are, the closer the shape is to being rectangular. The long side of the rectangle is made from half of the circumference of the circle which is $\frac{1}{2} \times \pi \times d = \pi \times r$. The short side of the rectangle is r and so the area of the rectangle (and therefore the circle) is $\pi \times r \times r$.

answer. The problems start when the students are later asked to find the percentage of the total background radiation that is contributed by cosmic rays (for example).

If the students *understood* percentages when they were considering energy loss from the house then this new context should be no problem, but if they have not understood then they need a whole new process that applies only to background radiation.

Exploring the language of mathematics

I find that a useful approach to many mathematical concepts is to explore the *language* of the maths in more detail. Breaking up the word percent into 'per' and 'cent', meaning 'for every' and 'one hundred' respectively, can be most instructive and using different but equivalent sequences of words can clarify the principle. 10 percent means 10 *for every 100*. 15 percent of the energy lost from the house is through the windows means that 15 joules of energy is lost from the windows *for every 100* joules lost from the house.

Even exploring the percent *sign* can be instructive. The mathematical shorthand for 'per' is the solidus (/). Students brought up in the internet age may refer to this symbol as the 'forward slash' but we should emphasise that the solidus is not simply a separator, which is the role of the forward slash in an internet URL. The percent sign comprises the two zero digits of the number 100 straddling the solidus and gives an insight into the meaning 'per 100'.

Students may well recognise that 'per' also appears in the units used (particularly) in physics. I find that this can be confused by the use of *mph* for miles per hour but in the modern system of units the solidus is again used to mean 'per' or 'for every'. Consider the following units and how replacing 'per' with 'for every' can be instructive in the meaning of the quantity being expressed:

> metres per second, joules per second, coulombs per second, joules per coulomb, kilograms per newton, newtons per square metre, and so on.

The first three of these express rates, meaning 'for every second'. It is important to understand that the universe is changing all the time and understanding the rate of change is central to physics. Understanding (or at least knowing) that speed, for example, is a rate of covering a certain distance is the first step towards understanding calculus, where rates can themselves be changing. Can a student really appreciate that acceleration is a rate of change of velocity without understanding first that velocity is a rate of change of distance?

The meaning behind formulae

Another use of the solidus is in the mathematical formulae that run through any physics syllabus. It could be argued that this is just the same as its use in units but for the sake of exemplification let us consider the formula relating to spring constants.

I find it unhelpful in this context to give the formula as $F = ke$, where F is the force applied to the spring, k is the spring constant and e is the extension. The formula looks much more like a tool for finding force than a mathematical explanation of how a spring behaves under tension. It looks like something to learn rather than something that could aid understanding.

It is more useful to give the formula as:

$$k = \frac{F}{e}$$

From this arrangement, we can explain that the spring constant tells us the force *per* metre of extension or, in the language of units, the number of newtons needed *for every* metre of extension.

Likewise, $F = ma$ is more instructive when expressed as

$$m = \frac{F}{a}$$

It actually defines inertial mass as the number of newtons needed to accelerate that mass for every metre per second per second; in effect the *resistance to acceleration* (Penrose, 2004: 392).

The use of the unit of acceleration as metre *per second per second* here is deliberate. In my experience, students are better able to understand that acceleration is the change in velocity per second, i.e. (m/s)/s, or even the change in distance covered per second *for every second* than to immediately employ the standard notation of m/s^2.

The reader might like to consider the following formulae, written intentionally in these arrangements, and think about how the language used above can help to improve students' understanding of the quantities they represent:

$$g = \frac{F}{m} \quad F = \frac{W}{d} \quad P = \frac{W}{t} \quad I = \frac{Q}{t} \quad V = \frac{W}{Q}$$

where g represents gravitational field strength, F represents force, W represents work done or energy transferred, m represents mass, d represents distance moved, P represents power, t represents time, I represents electrical current, Q represents charge transferred and V represents potential difference.

I am not suggesting that *all* formulae should be given in this form. When defining momentum as the product of the mass and the velocity it is usual to give the formula in the form $p = m \times v$. I cannot imagine that giving the students

$$m = \frac{p}{v}$$

to learn would be advantageous but this is because the momentum is defined as the mass 'times' the velocity. However, what advantage *for understanding* is there in giving the speed formula as *distance travelled* = *speed* × *time*, as it is in the June 2015 *Combined Science GCSE Subject Content* (Department for Education, 2015: 37)? Surely

$$speed = \frac{distance\ \ travelled}{time}$$

is more instructional?

Explaining the meaning behind the formulae should help students to learn them; the subject content for combined science in England gives a list of formulae that students should be able to recall (Department for Education, 2015: 37). The explanation should also help students to understand the formulae as well as help to reduce confusion brought about by formulae being presented in inconsistent ways. I will return to this point later.

A source of confusion relating to symbolism is that the horizontal line *dividing* the numerator and denominator can be thought of as 'divided by' as well as 'per'. Checking prior knowledge and understanding is central to teaching but how often do we, as teachers, check that students understand that Q/t is equivalent to $Q \div t$? Do we assume that students understand this when introducing formulae? Do we check that students are not confused when Q/t becomes 10/5 but that to find the answer they must carry out the operation $10 \div 5$?

The fundamental importance of 'is equal to'

Some thought about the students' prior knowledge and understanding of what we might consider to be basic operations and symbolism is as worthwhile an activity as is considering prior knowledge in any other context. A fundamental example is the students' 'understanding' of what the equals sign (=) represents.

We consider the equals sign to represent 'equals' and most students will say this when asked, but is their understanding of 'is equal to' demonstrated in their work? How often have we seen $4 \times 3 = 12 \div 6 = 2$ when students carry out multistep calculations? It either demonstrates a misunderstanding of the meaning of 'is equal to' or it demonstrates a disregard for its importance.

The following conversation with my 6-year-old son suggests that misconceptions about the meaning of the equals sign could well be formed at an early age:

Son: *Daddy, I know what 10 times 10 equals.*

Daddy: *That's great, son… tell me.*

Son: *100.*

Daddy: *Well done. What does equals mean?*

Son: *Makes.*

Had I not been writing this article at the time I would not have asked the question; I would probably have just explored his mathematical prowess further. His 'understanding' of equals as an operator (10 times 10 *makes* 100) suggests that the misconception that many secondary students hold could well have been formed many years earlier. (My wife teaches 6-year-olds and on hearing our conversation reported that she teaches her pupils 'is equal to' not 'makes'.)

Promoting the importance of the equals sign by limiting its use to one per line can help, as can encouraging students to read their equations out loud. An example of good practice is shown in Figure 3, which many teachers will recognise as being 'the way we used to do it'. Interestingly, a student who writes $4 \times 3 = 12 \div 6 = 2$ in a GCSE exam (taken by 16-year-olds in England) will score full marks for certain awarding bodies providing the final answer is correct and despite the fact that the mathematical grammar is not. Perhaps this means that there is little impetus for teachers to correct this type of mistake. I suggest that the fundamental misunderstanding that it implies means that teachers should correct it.

An understanding of the true meaning of the equals sign (as opposed to an alternative perception of the sign as an operator or as a

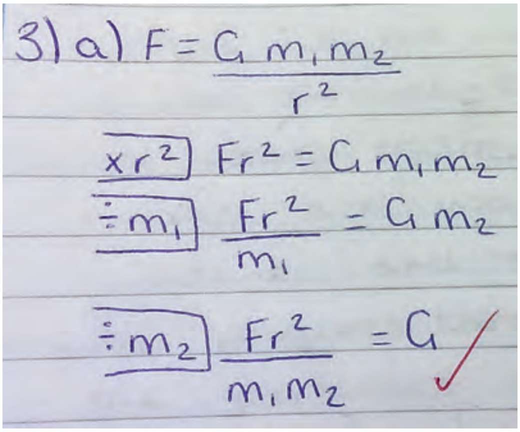

Figure 3 'The way we used to do it'? Here the student is noting what operation is being performed on each line of work. This helps to emphasise the importance of the equals sign.

symbol that simply separates calculations) is essential to students' understanding of algebraic manipulation as well as arithmetic procedures.

Rearranging simple equations

Rearranging equations is a key skill in physics and students who have not mastered the art at GCSE run into serious difficulties very early on in an A-level course.

Once again, it is essential to take into account the students' prior experiences of rearranging formulae. Some will have been taught processes that lead to no understanding of, or even reference to, the rules that apply to algebra. Two such examples are 'formula triangles' and 'the magic bridge' (where unknowns are allowed to move diagonally through the equals sign).

I argue that an understanding that allows students to transfer their methods between formulae, topics and subjects has to be based on an understanding of the equals sign. When we 'do something' to one side of a formula that we fail to do to the other side, the meaning of the equals sign is violated. This can only be exemplified by (at least in the early stages) doing the same thing to both sides of a formula or equation such that the equals sign maintains its fundamental meaning. This process always leads to one side of the formula having the same symbol or number in both the numerator and the denominator, whereupon students must understand that anything divided by itself is 1 (not zero!).

One way of stressing the importance of maintaining equality on both sides of an equation is to substitute some arbitrary values. Consider a simple example, $m = F \div a$, which we wish to rearrange so that F is the subject.

Multiply both sides by a so that $m \times a = F \div a \times a$.

Now substitute some arbitrary values, let's say $5 = 10 \div 2$ such that the rearrangement becomes $5 \times 2 = 10 \div 2 \times 2$.

Because the equals sign is so powerful and the left-hand side has become 5×2, the right-hand side must be equal to 10; the 2s have become redundant. If students choose the values themselves, they can see that this general rule works for any set of values.

The explicit use of × and ÷ signs above is intentional. If students do not understand the rules of algebraic manipulation in terms of multiplication and division, they are unlikely to understand (and therefore won't be able to transfer) the processes that they apply to rearranging a formula written as

$$m = \frac{F}{a}$$

I will always demonstrate formula work by writing out in full until I am absolutely sure that students understand the shorthand:

> $R = V \div I$ rather than $R = V/I$,
> and $P = V \times I$ rather than $P = VI$.

It is also worth looking carefully at the ways in which awarding bodies represent the formulae in different situations and ascertaining whether the students will recognise them or not. An example of inconsistency is given here from a draft GCSE physics specification where operators are sometimes included and sometimes not; symbols are also mixed with words:

> The equations $F = m \times a$ and $a = \frac{v-u}{t}$ lead to the equation $F = \frac{m\Delta v}{\Delta t}$ where $m\Delta v$ = change in momentum. (AQA, 2015: 32)

While students are unlikely to study the specification, I would check the formula sheet as well as the sample assessment materials for consistency and be sure that students understand them.

Demonstrate transferability

Most teachers will acknowledge that the curriculum is overloaded and that there is insufficient time to teach the depth or rigour that we would like. As a result, topics can be taught in isolation, thereby reinforcing the compartmentalisation that students do so naturally. I suggest that taking the time to demonstrate how certain skills are applicable across a range of topics can actually save time and help students to transfer their learning. Students will not do it naturally (in my experience) so they must be shown. During a lesson involving rate of change of momentum, take time to refer back to the lesson on acceleration where the rate of change of velocity was studied. Compare the formulae, compare the graphs and discuss the similarities, particularly in the nature of changes *per second*.

There are many other opportunities in the physics syllabus to refer back to situations or topics where certain skills have been applied before and this serves as good revision too.

Summary

I am sure that colleagues will recognise most, if not all, of the problems discussed so far and I hope that putting into practice what I have suggested will help students to understand their mathematical work better in the sense that they can transfer their understanding between topics.

New problems will be encountered with the mathematical content of the new GCSEs; in particular, the statements that students will be required to '*make estimates*' and '*make order of magnitude calculations*' (Department for Education, 2015: 40) give us no clue as to what context or level of difficulty these might be. I hope that applying the principles summarised in Box 1 will help.

BOX 1 Summary of key points

In order to help students better understand the mathematical content of their science and be able to transfer skills across subjects we should:

- check prior knowledge and understanding of notations, symbols and conventions;
- demonstrate and provide opportunities for students to experience the reasons why mathematical processes are the way they are;
- demonstrate and provide opportunities for students to experience the meaning behind mathematical formulae;
- highlight the importance of units and how they reflect the way that algebraic processes work;
- promote the use of one equals sign per line of calculations;
- demonstrate and encourage algebra written out longhand until the students are secure with the shorthand;
- explore the language behind the mathematics;
- refer back to topics where the same mathematical processes have already been learned;
- check formula sheets for consistency and ensure that students understand the different ways that formulae can be presented to them.

Acknowledgement

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